

PERFORMANCE EVALUATION CAPABILITIES FOR THE
DESIGN OF PHYSICAL SYSTEMS

Final Report on
NASA Grant No. NGR 47-005-145
Supplement No.1

September
1972

SUBMITTED BY

W.D. Pilkey
B.P. Wang



Division of Aerospace Engineering and
Engineering Physics
Research Laboratories for Engineering Sciences
University of Virginia
Charlottesville, Virginia

Report No. AEEP-4085-107-72U

(NASA-CR-112293) PERFORMANCE EVALUATION
CAPABILITIES FOR THE DESIGN OF PHYSICAL
SYSTEMS Final Report (Virginia Univ.)
25 p HC \$3.25

CSSL 12B

N73-20261

Unclas

G3/10 67592

FOREWORD

The research described in this report (University of Virginia Project No. 4085-1320) was performed at the University of Virginia Research Laboratories for the Engineering Sciences. W.D. Pilkey was principal investigator. The project was performed under NASA Grant No. NGR 47-005-145, Supplement Number 1 and administered at NASA Langley Research Center. This report covers the work performed during the second year of this grant. The continued assistance of the technical monitors for this study, J. Sewall and R. Parrish of NASA/Langley, is appreciated.

The cooperation of Murray Manufacturing of Earlysville, Virginia in permitting the use of their computer terminal for this project is gratefully acknowledged.

ABSTRACT

This report documents the accomplishments of the NASA sponsored effort to develop performance evaluation capabilities for the design of physical systems. These accomplishments are:

- . Development of a theory of limiting performance of large systems subject to steady state inputs
- . Application and modification of PERFORM, the computational capability for the limiting performance of systems with transient inputs
- . Demonstration that use of an inherently smooth control force for a limiting performance calculation improves the system identification phase of the design process for physical systems subjected to transient loading.

INTRODUCTION

The primary goals of this study were to formulate and develop a capability for the limiting performance of large steady state systems. It was shown early in the study that the previously developed capability for transient systems could not be extended to steady state environments. It was necessary to begin anew and formulate a different theory of limiting performance. This new theory has been applied to several simple systems. Most of this report deals with the formulation of the limiting performance problem for steady state systems.

LIMITING PERFORMANCE OF STEADY STATE SYSTEMS

The concept and importance of limiting performance of mechanical systems is described in Ref. 1.

The formulation of the limiting performance study of a mechanical system subject to steady state sinusoidal loading is given in Appendix IA. The formulation is in terms of a linear programming problem. The underlying concept in this formulation is the use of a Fourier expansion to represent both the responses and control forces. The coefficients in the Fourier expansion of control forces are the variables to be found. This formulation is applied to a single degree of freedom (SDF) system for both linear and nonlinear control forces in Appendix IB. The solution for the linear case corresponds to results in Ref. 1, while for the general case the solution compares well with results obtained by an optimal control approach (Ref. 2).

The formulation of Appendix IA is time-dependent. For a linear system the problem is reformulated in Appendix IIA as a time-independent nonlinear programming problem. The merit of this formulation is its time-independence with a concomitant reduction in the dimension of the problem. The advantage of obtaining the limiting performance of a system without multiple analyses of the system has been retained.

This formulation is applied to two special cases in Appendix IIB. In both cases the results agree with known solutions.

The steady state limiting performance solution is extended to systems with multiple forcing functions of different frequencies in Appendix III.

The major remaining unresolved problem for the steady state formulation is that of limiting performance of systems subject to forcing functions with frequencies varying over a prescribed range. The present formulations are restricted to systems subjected to loading at a prescribed frequency. The study of the frequency range problem is continuing.

APPLICATION AND MODIFICATIONS OF PERFORM

During the first year of this effort a computer capability (PERFORM) was developed for calculating the limiting performance of systems with transient inputs. The final report covering this effort contains detailed documentation and applications of PERFORM. This report has been revised and is now entitled PERFORM - A PERFORMANCE OPTIMIZING COMPUTER PROGRAM FOR DYNAMIC SYSTEMS SUBJECT TO TRANSIENT LOADINGS. The revised report contains new results for a train impact problem and a STOL ride control problem. The application to the train impact problem was used as part of the paper "Limiting Performance of Ground Transportation Vehicles Subject to Transient Loading" (Ref. 3) presented at the AIAA/ASME/ASCE 13th Structures, Structural Dynamics, and Materials Conference. In the problem, the train impact model from Ref. 4 is modified so that the cushion or shock absorber is replaced by a control force. The problem is to find the minimum force that must be transmitted to the lading for a given cushion travel distance under specified impact conditions. Performance tradeoff diagrams were computed. Details of the problem formulation are given in Ref. 3.

The limiting performance problem of the ride control system for the Twin-Otter STOL airplane (Ref. 5) has been put in PERFORM format. Computations of the sort shown in Fig. 1 were made. Here the tradeoff between the min-max acceleration at the center of gravity of the airplane and the level of controls was calculated using PERFORM. Details of the formulation are to be found in Ref. 5.

PERFORM has been modified to permit its use in conjunction with OPTIMA, a CDC computer software system that can solve large linear programming problems. Previously, the IBM program MPS/360 was used.

SYSTEM IDENTIFICATION

The use of a Fourier series expansion of control forces in the steady state problem suggested a possible application for the design of transient systems. It was shown that use of the "smooth" control force employed in the steady state problem for the limiting performance of transient systems eases the task of identifying the corresponding suboptimal design configuration. Heretofore, transient systems had been designed using the limiting performance based on control forces that were given a piecewise constant time discretization. This application of the Fourier series to limiting performance problems was demonstrated for single and two degree of freedom systems.

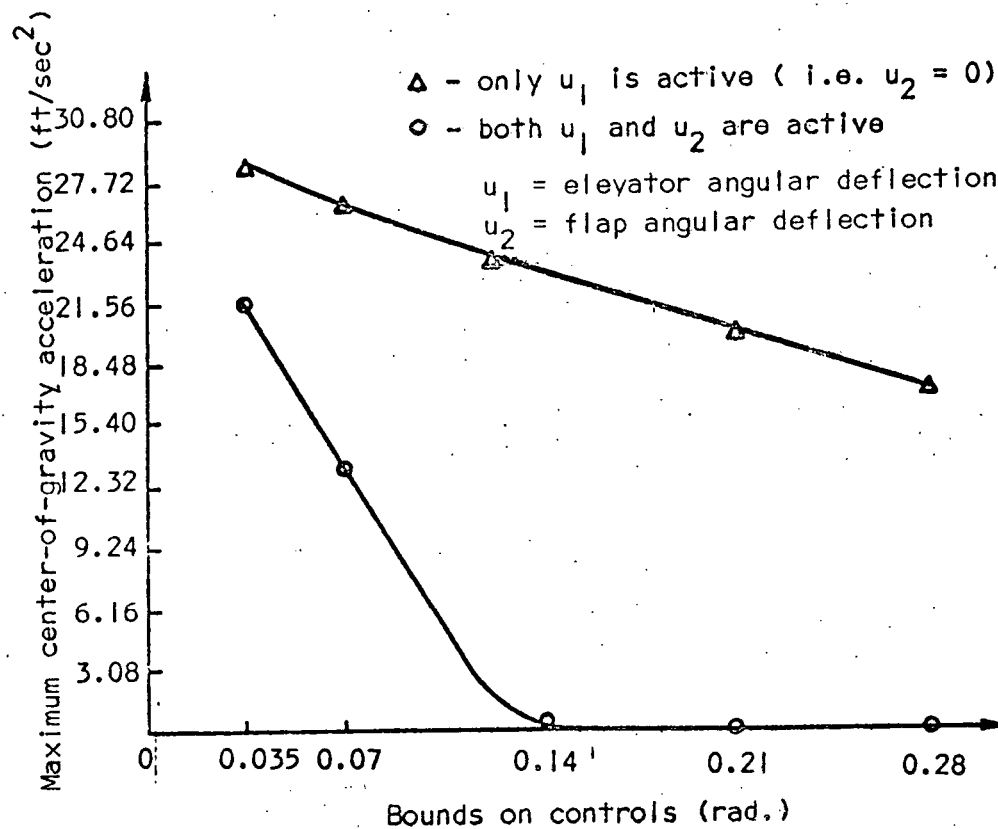


Fig . 1 Limiting Performance of Twin-Otter Airplane Ride Control System

REFERENCES

1. Sevin, E. and Pilkey, W. Optimal Shock and Vibration Isolation, Shock and Vibration Information Center, Washington, D.C., 1971.
2. Parrish, R., NASA/Langley, Private Communication, 1972.
3. Pilkey, W.D. and Wang, B.P. "Limiting Performance of Ground Transportation Vehicles Subject to Transient Loading", AIAA Paper No. 72-340, 1972.
4. Peterson, W.H. "Cushioning Requirements for Adequate Lading Protection", ASME Paper No. 59-A-312, 1959.
5. Leatherwood, J.D., NASA/Langley, Private Communication, 1972.
6. Den Hartog, J.P. Mechanical Vibrations, Fourth Edition, McGraw Hill, 1956.

APPENDIX 1A

Formulation of the Limiting Performance Problem for a Dynamic System Subject to Steady-State Sinusoidal Disturbances.

Consider a dynamic system described by the following equations of motion:

$$\underline{\ddot{M}}\underline{\ddot{X}} + \underline{\dot{C}}\underline{\dot{X}} + \underline{K}\underline{X} + \underline{V}\underline{U} = \underline{F}\underline{\bar{f}} \quad (1)$$

where

$\underline{M} = N \times N$	mass matrix
$\underline{C} = N \times N$	damping matrix
$\underline{K} = N \times N$	stiffness matrix
$\underline{V} = N \times J$	coefficient matrix associated with control force vector
$\underline{F} = N \times L$	coefficient matrix associated with forcing function vector

and

N	= no. of degrees of freedom of the system
J	= no. of isolators or controllers
L	= no. of disturbances applied on the system

Let

$$\underline{\bar{f}} = \begin{bmatrix} f_1 \sin \omega t \\ f_2 \sin \omega t \\ \vdots \\ f_L \sin \omega t \end{bmatrix} = \underline{\bar{f}}_0 \sin \omega t \quad (2)$$

where

$$\bar{f}_0 = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_L \end{bmatrix}$$

Assume

$$\bar{U} = \sum_{i=1}^m \left[\bar{U}_{si} \sin i\omega t + \bar{U}_{ci} \cos i\omega t \right] \quad (3)$$

where

$$\bar{U}_{si} = \begin{bmatrix} U_{1si} \\ U_{2si} \\ \vdots \\ U_{Jsi} \end{bmatrix} \quad \bar{U}_{ci} = \begin{bmatrix} U_{1ci} \\ U_{2ci} \\ \vdots \\ U_{Jci} \end{bmatrix} \quad (3a)$$

m = number of pairs of terms used in the Fourier representation of \bar{U}

Similarly, assume

$$\bar{X} = \sum_{i=1}^m \left[\bar{X}_{si} \sin i\omega t + \bar{X}_{ci} \cos i\omega t \right] \quad (4)$$

where

$$\bar{X}_{si} = \begin{bmatrix} X_{1si} \\ X_{2si} \\ \vdots \\ X_{Nsi} \end{bmatrix} \quad \bar{X}_{ci} = \begin{bmatrix} X_{1ci} \\ X_{2ci} \\ \vdots \\ X_{Nci} \end{bmatrix}$$

Define

$$\bar{X}_i = \begin{bmatrix} \bar{X}_{si} \\ \text{---} \\ \bar{X}_{ci} \end{bmatrix} \quad \bar{U}_i = \begin{bmatrix} \bar{U}_{si} \\ \text{---} \\ \bar{U}_{ci} \end{bmatrix} \quad (5)$$

Substitute (3), (4) into (1) and use (5) to obtain

$$\bar{X}_i = \underline{W}_i \bar{U}_i + \underline{Z}_i \bar{f}_0 \quad (6)$$

where

$$\underline{W}_i = \underline{Y}_i^{-1} \begin{bmatrix} -\underline{V} & \underline{0} \\ \text{---} & \text{---} \\ \underline{0} & -\underline{V} \end{bmatrix} = \begin{bmatrix} \underline{WSS}_i & \underline{WSC}_i \\ \text{---} & \text{---} \\ \underline{WCS}_i & \underline{WCC}_i \end{bmatrix}$$

$$\underline{Z}_i = \underline{Y}_i^{-1} \begin{bmatrix} \delta_{li} \underline{F} \\ \text{---} \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{ZS}_i \\ \text{---} \\ \underline{ZC}_i \end{bmatrix} \quad (7)$$

$$\delta_{li} = \begin{cases} 1 & \text{if } i = l \\ 0 & \text{if } i \neq l \end{cases}$$

$$\underline{Y}_i = \begin{bmatrix} -\underline{M}\omega^2 i^2 + \underline{K} & -\underline{C}i\omega \\ \text{---} & \text{---} \\ \underline{C}i\omega & -\underline{M}\omega^2 i^2 + \underline{K} \end{bmatrix}$$

Thus, from equation (6), all the coefficients in the Fourier expansion of the responses (i.e. X's) can be expressed in terms of those of U's.

From Eqs. (6) and (4) we get

$$\bar{X} = \underline{R}_m \bar{U}_m + \underline{T}_m \bar{f}_0 \quad (8)$$

where

$$\underline{R}_m = \begin{bmatrix} \underline{WSS}_1 \sin \omega t + \underline{WCS}_1 \cos \omega t & \underline{WSC}_1 \sin \omega t + \underline{WCC}_1 \cos \omega t \\ \dots \dots \underline{WSS}_m \sin m \omega t + \underline{WCS}_m \cos m \omega t & \underline{WSC}_m \sin m \omega t + \underline{WCC}_m \cos m \omega t \end{bmatrix}$$

$$\underline{T}_m = \sum_{i=1}^m \left[\underline{ZS}_i \sin i \omega t + \underline{ZC}_i \cos i \omega t \right] \quad (9)$$

$$\bar{\underline{U}}_m = \begin{bmatrix} \bar{\underline{U}}_1 \\ \vdots \\ \bar{\underline{U}}_m \end{bmatrix} = \begin{bmatrix} \bar{\underline{U}}_{s1} \\ \bar{\underline{U}}_{c1} \\ \vdots \\ \bar{\underline{U}}_{sm} \\ \bar{\underline{U}}_{cm} \end{bmatrix}$$

and from (3)

$$\bar{\underline{U}} = \underline{SC}_m \bar{\underline{U}}_m \quad (10)$$

where

$$\underline{SC}_m = \begin{bmatrix} \underline{I} \sin \omega t & \underline{I} \cos \omega t & \dots \dots \underline{I} \sin m \omega t & \underline{I} \cos m \omega t \end{bmatrix}$$

$\underline{I} = J \times J$ Identity matrix

We have expressed $\bar{\underline{X}}, \bar{\underline{U}}$ in terms of $\bar{\underline{U}}_m$, a set of unknown numbers. Note that the matrices $\underline{T}_m, \underline{R}_m, \underline{SC}_m$ are all functions of time.

Suppose the limiting performance problem of interest can be stated as: Find the $\bar{\underline{U}}$ that minimizes the maximum $\bar{\Psi}$ while subject to the constraints

$$Y L_i \leq \theta_i \leq Y U_i$$

where

$$\bar{\Psi} = \underline{P1} \ddot{\bar{\underline{X}}} + \underline{P2} \dot{\bar{\underline{X}}} + \underline{P3} \bar{\underline{X}} + \underline{P4} \bar{\underline{U}} + \underline{P5} \bar{\underline{f}} \quad (11)$$

$$\bar{\theta} = \underline{Y1} \ddot{\bar{X}} + \underline{Y2} \dot{\bar{X}} + \underline{Y3} \bar{X} + \underline{Y4} \bar{U} + \underline{Y5} \bar{f} \quad (12)$$

This problem can be converted to: Find ϕ, \bar{U} such that ϕ is minimized subject to

$$\begin{aligned} |\psi_j| &\leq \phi & \text{for } j = 1, 2, \dots, \text{NOB} \\ \underline{Y}L_i &\leq \theta_i \leq \underline{Y}U_i & \text{for } i = 1, 2, \dots, \text{NOC} \end{aligned} \quad (13)$$

where

NOB = number of objective functions

NOC = number of constraints

The problem defined by (13) is one of linear programming, i.e.

Find \bar{z}

to minimize $\underline{c} \bar{z}$

subject to $\underline{H} \bar{z} \begin{matrix} \geq \\ \leq \end{matrix} \bar{G}$

where

$$\bar{z} = \begin{bmatrix} \phi \\ \bar{U}_m \end{bmatrix} \quad \underline{c} = [1, 0, 0, \dots, 0]$$

$$H = \begin{bmatrix} -\bar{I} & \underline{RG}_m \\ +\bar{I} & \underline{RG}_m \\ \bar{0} & \underline{RY}_m \\ \bar{0} & \underline{RY}_m \end{bmatrix}$$

$$G = \begin{bmatrix} -\underline{TG}_m \bar{f}_0 \\ -\underline{TG}_m \bar{f}_0 \\ \underline{YU} - \underline{TY}_m \bar{f}_0 \\ \underline{YL} - \underline{TY}_m \bar{f}_0 \end{bmatrix}$$

where

$$\underline{RG}_m = \underline{P1} \underline{R2}_m + \underline{P2} \underline{R1}_m + \underline{P3} \underline{R}_m + \underline{P4} \underline{SC}_m$$

$$\underline{TG}_m = \underline{P1T2}_m + \underline{P2T1}_m + \underline{P3T}_m + \underline{P5\bar{f}}_O \sin \omega t$$

$$\underline{RY}_m = \underline{Y1R2}_m + \underline{Y2R1}_m + \underline{Y3R}_m + \underline{Y4SC}_m$$

$$\underline{TY}_m = \underline{Y1T2}_m + \underline{Y2T1}_m + \underline{Y3T}_m + \underline{Y5\bar{f}}_O \sin \omega t$$

Here \underline{R}_m , \underline{T}_m , are functions of time and

$$\underline{R2}_m = \ddot{\underline{R}}_m(t)$$

$$\underline{R1}_m = \dot{\underline{R}}_m(t)$$

$$\underline{T2}_m = \ddot{\underline{T}}_m(t)$$

$$\underline{T1}_m = \dot{\underline{T}}_m(t)$$

APPENDIX 1B

Solutions of Limiting Performance Problem for Single Degree of Freedom System

Consider the SDF problem (Fig. 1-1) of finding U such that $\phi = \max|\ddot{Z}| = \max|U|$ is minimized while $|X| \leq A$

The equation of motion is

$$m\ddot{Z} + U = 0$$

$$Z = X + f = X + \sin\omega t$$

These give

$$\ddot{X} + U = \omega^2 \sin\omega t$$

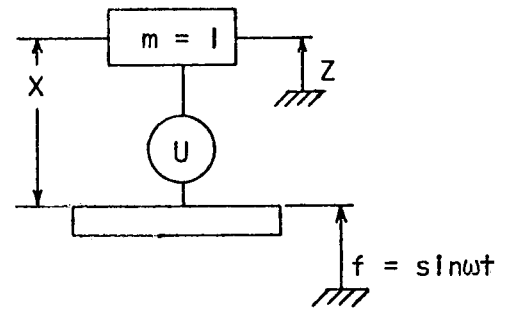


Figure 1-1

for a unit mass.

For $A = 0.5$, $\omega = 20$, the optimal control approach (Ref. 2) gives $\min\phi = 162.4$. The same problem has been solved by linear programming using 9 pairs of terms in a Fourier representation of U , and the result is $\min\phi = 163.2$. These solutions are very close. The optimal control approach is practical for simple systems only, while the linear programming formulation applies to large systems.

For the first time in limiting performance studies it is possible to distinguish between linear and nonlinear control forces. If the Fourier series is restricted to one sine and one cosine term then the performance of a linear system is obtained. For the linear SDF problem, the linear programming formulation provides the same solution calculated for this case in Ref. 1. The new formulation is significant because the treatment of Ref. 1 cannot be extended to complex systems to which the linear programming formulation can be applied.

APPENDIX IIA

Formulation of a Time-Independent Limiting Performance Solution for Linear Dynamic Systems Subject to Sinusoidal Inputs

Consider a multidegree of freedom system described by

$$\underline{M}\ddot{\underline{X}} + \underline{C}\dot{\underline{X}} + \underline{K}\underline{X} + \underline{V}\underline{U} = \underline{F}\bar{f}_0 \sin \omega t \quad (1)$$

Since we seek the limiting performance for a linear system, let

$$\underline{U} = \underline{U}_s \sin \omega t + \underline{U}_c \cos \omega t \quad (2)$$

$$\underline{X} = \underline{X}_s \sin \omega t + \underline{X}_c \cos \omega t$$

Substitute (2) into (1) to give

$$\begin{bmatrix} \underline{X}_s \\ \underline{X}_c \end{bmatrix} = \underline{W} \begin{bmatrix} \underline{U}_s \\ \underline{U}_c \end{bmatrix} + \underline{Z}\bar{f}_0 \quad (3)$$

where

$$\underline{W} = \begin{bmatrix} -\underline{M}\omega^2 + \underline{K} & -\underline{C}\omega \\ \underline{C}\omega & -\underline{M}\omega^2 + \underline{K} \end{bmatrix}^{-1} \begin{bmatrix} -\underline{V} & \underline{0} \\ \underline{0} & -\underline{V} \end{bmatrix} = \begin{bmatrix} \underline{WSS} & \underline{WSC} \\ \underline{WCS} & \underline{WCC} \end{bmatrix}$$

$$\underline{Z} = \begin{bmatrix} -\underline{M}\omega^2 + \underline{K} & -\underline{C}\omega \\ \underline{C}\omega & -\underline{M}\omega^2 + \underline{K} \end{bmatrix}^{-1} \begin{bmatrix} \underline{F} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{ZS} \\ \underline{ZC} \end{bmatrix}$$

Let the objective function be

$$\bar{\Psi} = \underline{P1}\ddot{\underline{X}} + \underline{P2}\dot{\underline{X}} + \underline{P3}\underline{X} + \underline{P4}\underline{U} + \underline{P5}\bar{f} \quad (4)$$

and the constraint be

$$\underline{Y1}\ddot{\underline{X}} + \underline{Y2}\dot{\underline{X}} + \underline{Y3}\underline{X} + \underline{Y4}\underline{U} + \underline{Y5}\bar{f} \leq \underline{YU} \quad (5)$$

Using (2) and (3), (4) and (5) become

$$\bar{\psi} = \bar{\phi}_s \sin \omega t + \bar{\phi}_c \cos \omega t \quad (6)$$

and

$$\bar{Y}L \leq \bar{\theta}_s \sin \omega t + \bar{\theta}_c \cos \omega t \leq \bar{Y}U \quad (7)$$

where

$$\begin{aligned} \bar{\phi}_s = & \frac{-P1}{\omega^2} \frac{WSS}{\bar{U}_s} \bar{U}_s \omega^2 - \frac{P1}{\omega^2} \frac{WSC}{\bar{U}_c} \bar{U}_c \omega^2 - \frac{P1}{\omega^2} \frac{ZS}{\bar{f}_0} \bar{f}_0 \omega^2 \\ & - \frac{P2}{\omega} \frac{WCS}{\bar{U}_s} \bar{U}_s \omega - \frac{P2}{\omega} \frac{WCC}{\bar{U}_c} \bar{U}_c \omega + \frac{P3}{\omega} \frac{WSS}{\bar{U}_s} \bar{U}_s \\ & + \frac{P3}{\omega} \frac{WSC}{\bar{U}_c} \bar{U}_c + \frac{P3}{\omega} \frac{ZS}{\bar{f}_0} \bar{f}_0 + \frac{P4}{\omega} \bar{U}_s + \frac{P5}{\omega} \bar{f}_0 \end{aligned} \quad (8)$$

The quantity $\bar{\phi}_c$ is the same as $\bar{\phi}_s$ with $\frac{WSC}{\omega}$, $\frac{WCC}{\omega}$ replacing $\frac{WSS}{\omega}$, $\frac{WSC}{\omega}$ respectively. $\bar{\theta}_s$ and $\bar{\theta}_c$ are the same as $\bar{\phi}_s$, $\bar{\phi}_c$ with $\frac{P1}{\omega}$ $\frac{P5}{\omega}$ replaced by $\frac{Y1}{\omega}$ $\frac{Y5}{\omega}$.

From (6), we can write

$$\max \psi_i = \phi_{si}^2 + \phi_{ci}^2 \quad (9)$$

and (7) is equivalent to

$$\theta_{si}^2 + \theta_{ci}^2 \leq YU_i^2 \quad (10)$$

since, in general, $YL_i = -YU_i$

Hence, if we define

$$\phi = \max_i [\phi_{si}^2 + \phi_{ci}^2] \text{ for } i = 1, 2, \dots, \text{NOB}$$

the limiting performance problem can be restated as: Find ϕ , \bar{U}_s , \bar{U}_c such that

ϕ is minimized and

$$\phi_{si}^2 + \phi_{ci}^2 \leq \phi \text{ for } i = 1, 2, \dots, \text{NOB}$$

and

$$\theta_{sj}^2 + \theta_{cj}^2 \leq YU_j^2 \text{ for } j = 1, 2, \dots, \text{NOC} \quad (11)$$

where NOB = no. of objective functions

NOC = no. of constraints

Using (8), everything in (11) can be expressed in terms of \bar{U}_s and \bar{U}_c . Note that the time dependency of the response variables has been eliminated. The problem posed by (11) is one of nonlinear programming.

APPENDIX IIB

Examples of Limiting Performance Bounds Based on the Time-Independent Formulation

1. SDF system (Fig. 11-1)

The equations of motion are

$$m\ddot{Z} + U = 0$$

$$Z = X + f = X + f_0 \sin \omega t$$

If $m = 1$

$$\ddot{X} + U = \omega^2 f_0 \sin \omega t$$

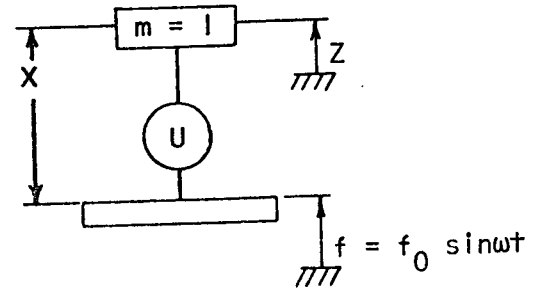


Figure 11-1 (1)

We want to find U that minimizes $|U|$ subject to $|X| \leq A$

Let

$$U = U_s \sin \omega t + U_c \cos \omega t \quad (2)$$

$$X = X_s \sin \omega t + X_c \cos \omega t$$

Then, following the previous formulation,

$$X_s = \frac{U_s}{\omega^2} - f_0 \quad X_c = \frac{U_c}{\omega^2}$$

The problem now is to find U that minimizes $(U_s^2 + U_c^2)$ subject to

$$(X_s^2 + X_c^2) = \left(\frac{U_s}{\omega^2} - f_0 \right)^2 + \left(\frac{U_c}{\omega^2} \right)^2 \leq A^2 \quad (3)$$

The solution can be found easily (e.g. by graphical means).

We get

$$U_s = \omega^2 (f_0 - A) \quad (4)$$

$$U_c = 0.0$$

$$\max |U| = \omega^2 (f_0 - A)$$

Equation (4), when normalized, is the same as the one obtained in Ref. 1.

2. Two Degree of Freedom (DOF) System

For the two DOF system shown in Fig. 11-2a, Den Hartog (Ref. 6) derived the result that under certain conditions, the main mass M in the system does not move at all.

This optimal condition can be obtained using the limiting performance approach. The equations of motion are (Fig. 11-2b)

$$M\ddot{X}_1 + KX_1 + U = P_0 \sin \omega t$$

$$m\ddot{X}_2 - U = 0$$

Then

$$U = U_s \sin \omega t + U_c \cos \omega t$$

$$X_i = X_{is} \sin \omega t + X_{ic} \cos \omega t \quad i = 1, 2$$

$$X_{1s} = \frac{U_s - P_0}{M\omega^2 - K} \quad X_{2s} = \frac{-U_s}{m\omega^2}$$

$$X_{1c} = \frac{U_c}{M\omega^2 - K} \quad X_{2c} = \frac{-U_c}{m\omega^2}$$

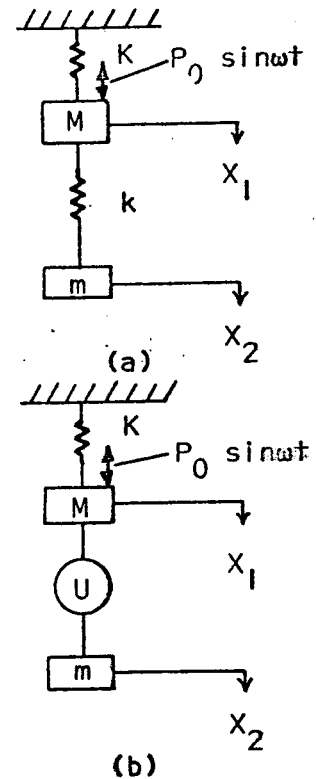


Figure 11-2

Now, we want to find the U that minimizes $|X_1|$. This is equivalent to finding the U that minimizes

$$X_{1s}^2 + X_{1c}^2 = \left[\frac{U_s - P_0}{M\omega^2 - K} \right]^2 + \left[\frac{U_c}{M\omega^2 - K} \right]^2$$

The solution is seen to be

$$U_s = P_0, U_c = 0 \quad \text{and} \quad \min \max |X_1| = 0$$

Suppose now we modified the problem by putting a constraint on the rattle space between the two masses. Now we seek $\min \max |X_1|$ while $|X_1 - X_2| \leq A$. Stated in terms of a nonlinear programming problem, this becomes: Find U_s, U_c to minimize

$$X_{1s}^2 + X_{1c}^2$$

subject to

$$(X_{1s} - X_{2s})^2 + (X_{1c} - X_{2c})^2 = A^2$$

The solution (valid for the case $M\omega^2 > K$) is

$$U_s = P_1 + \gamma_1, U_c = 0$$

$$\min \max |X_1| = \frac{P_1 + \gamma_1 - P_0}{|M\omega^2 - K|}$$

where

$$P_1 = \frac{P_0 m \omega^2}{-M\omega^2 + K + m\omega^2}$$

$$\gamma_1 = \left| \frac{m\omega^2 (M\omega^2 - K)}{M\omega^2 - K - m\omega^2} \right| A$$

This problem would be very difficult if it were to be treated using the Den Hartog approach.

where

$$\bar{x}_{jpl} = \begin{bmatrix} x_{1jpl} \\ x_{2jpl} \\ \vdots \\ x_{Njpl} \end{bmatrix} \quad \text{for } p = s, c$$

Substitute (2), (3) into (1) and equate the coefficients of terms $\cos i\omega_j t$ and $\sin i\omega_j t$ respectively. Then

$$(-Mi^2\omega_j^2 + K) \bar{x}_{jsi} - (Ci\omega_j) \bar{x}_{jci} = \bar{F}_j \delta_{ji} - \underline{v} \bar{U}_{jsi}$$

$$(Ci\omega_j) \bar{x}_{jsi} - (-Mi^2\omega_j^2 + K) \bar{x}_{jci} = -\underline{v} \bar{U}_{jci}$$

$$\text{where } \bar{F}_j = \underline{F} \begin{bmatrix} 0 \\ \vdots \\ f_j \\ \vdots \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} \bar{x}_{jsi} \\ \bar{x}_{jci} \end{bmatrix} = \begin{bmatrix} \underline{WSS}^{ji} & \underline{WSC}^{ji} \\ \underline{WCS}^{ji} & \underline{WCC}^{ji} \end{bmatrix} \begin{bmatrix} \bar{U}_{jsi} \\ \bar{U}_{jci} \end{bmatrix} + \begin{bmatrix} \underline{ZS}^{ji} \\ \underline{ZC}^{ji} \end{bmatrix} \quad (4)$$

with

$$\underline{W}^{ji} = \begin{bmatrix} \underline{WSS}^{ji} & \underline{WSC}^{ji} \\ \underline{WCS}^{ji} & \underline{WCC}^{ji} \end{bmatrix} = \underline{Y}^{ij} \begin{bmatrix} -\underline{v} & 0 \\ 0 & -\underline{v} \end{bmatrix}$$

and

$$\bar{Z}^{ji} = \begin{bmatrix} \bar{ZS}^{ji} \\ \bar{ZC}^{ji} \end{bmatrix} = \underline{Y}^{ij} \begin{bmatrix} \bar{F}_j \delta_{ii} \\ 0 \end{bmatrix}$$

$$\underline{Y}^{ij} = \begin{bmatrix} -\underline{M}i^2\omega_j^2 + \underline{K} & -\underline{C}i\omega_j \\ \underline{C}i\omega_j & -\underline{M}i^2\omega_j^2 + \underline{K} \end{bmatrix}^{-1}$$

From (3) and (4) we have

$$\bar{X} = \sum_{j=1}^L \sum_{i=1}^m \left[\underline{R}_m^{ji} \bar{U}_m^{ji} + \bar{T}_m^{ji} \right] \quad (5)$$

where

$$\underline{R}_m^{ji} = \begin{bmatrix} \underline{WSS}^{ji} \sin i\omega_j t + \underline{WCS}^{ji} \cos i\omega_j t \\ \vdots \\ \underline{WSC}^{ji} \sin i\omega_j t + \underline{WCC}^{ji} \cos i\omega_j t \end{bmatrix}$$

$$\bar{T}_m^{ji} = \begin{bmatrix} \bar{ZS}^{ji} \sin i\omega_j t + \bar{ZC}^{ji} \cos i\omega_j t \end{bmatrix}$$

$$\bar{U}_m^{ji} = \begin{bmatrix} \bar{U}_{jsi} \\ \bar{U}_{jci} \end{bmatrix}$$

Equation (2) can be rewritten as

$$\bar{U} = \sum_{j=1}^L \sum_{i=1}^m \underline{SC}_m^{ji} \bar{U}_m^{ji} \quad (6)$$

where

$$\underline{SC}_m^{ji} = \begin{bmatrix} \underline{I} \sin i\omega_j t \\ \vdots \\ \underline{I} \cos i\omega_j t \end{bmatrix}$$

\underline{I} is a $J \times J$ identity matrix.

We have just expressed \bar{X} and \bar{U} in terms of the unknown coefficients \bar{U}_m^{ji} . Following the procedure in Appendix IA, the limiting performance problem can now be placed in linear programming form with the coefficients of \bar{U} as unknowns.